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Journal of Electrostatics 61 (2004) 107–115
Journal of ELECTROSTATICS
www.elsevier.com/locate/elstat

Electrostatic force acting on a spherically symmetric charge distribution in contact with a conductive plane

W. Stanley Czarnecki[a,b,1], L.B. Schein[a,c,*]

[a] Aetas Technology Corporation, P.O. Box 53398, Irvine, CA 92619, USA
[b] Torrey Pines Research, 6359 Paseo Del lago, Carlsbad, CA 92009, USA
[c] 7026 Calcaterra Dr., San Jose, CA 95120 USA

Received 7 November 2003; accepted 25 January 2004

Abstract

A spherically symmetric charge distribution is a useful, first-order model of a charged insulating particle, such as a toner particle used in electrostatic imaging technologies. The electrostatic force of adhesion between a spherically symmetric charge distribution in contact with a conductive ground plane is derived using the method of images. Finite element analysis is used, with the uniformly charged sphere of charge Q modeled with K charge points, each of which has a charge of magnitude Q/K. Both a closed form solution and a numerical calculation are used to derive the total force. It is found that the electrostatic image force acting on the few charge points located in proximity to the conductive plane (the proximity charge) is comparable to the electrostatic force acting on a single charge point of magnitude Q located in the center of the sphere. This is a surprising result since it is conventionally assumed that a spherically symmetric charge distribution can be modeled by a single charge point Q located in the center of the sphere. However, this assumption is only valid at distances far from any conductive planes. When all pairs of charge points and their charge images are considered, the total electrostatic force acting on a sphere of charge Q in contact with a conductive plane is larger than the conventionally assumed electrostatic image force by a correction factor $(1 + 4/\pi)$. The additional force, that we call the proximity force which is primarily due to the proximity charge (the $4/\pi$ term), is independent of the number of K charge points assumed, suggesting it is model independent at contact. The proximity force's functional dependence on the separation between the bottom of the charged sphere and the conductive plane depends on

*Corresponding author. Tel./fax: +1-408-997-7946.
E-mail address: schein@prodigy.net (L.B. Schein).
[1] Current address: IBM Corporation, 5600 Cottle Rd., Building 13, San Jose, CA 95193, USA.

0304-3886/$ - see front matter
doi:10.1016/j.elstat.2004.01.022

the distribution of the charge points near the contact point. Implications of the existence of the electrostatic proximity force are discussed.

Keywords: Adhesion; Electrostatic adhesion; Toner; Particle; Toner adhesion; Imaging; Electrophotography; Proximity force; Atomic force microscopy

1. Introduction

Understanding the electrostatic force of adhesion acting on a charged insulating sphere in contact with a conductive plane has important applications in many different fields such as electrophotography, semiconductors, atomic force microscopy, and micro-tribology. For example, in electrophotography, a tribolectrically charged toner particle must be efficiently developed on a photoreceptor and transferred to paper using electric fields [1]. The Coulomb force (the charge times the electric field) must overcome the toner adhesion [2,3]. In the fabrication of semiconductor devices, removal of electrostatically adhering micron-sized contaminants can be critical to their functionality. In studies with the atomic force microscope unexpected adhesion forces have been observed in force–displacement curves [4–6]. Adhesion forces between materials during contact and rubbing can be caused by the buildup of charge on either surface.

The purpose of this paper is to examine a theoretical model of the adhesion of a uniformity charged sphere to a conductive plane, one of the components of adhesion. People often assume that a spherically symmetric charge distribution can be equivalently replaced with a single point charge in the center of the sphere. This is true only in the case of an isolated sphere. It relies on spherical symmetry to apply Gauss' Law. However, in the situation in which the spherically symmetric charge distribution is in contact with a conductive plane, the spherical symmetry no longer exists and no simple integral can be found to apply Gauss's Law. Since the conductive plane is an equipotential, the method of images can be used. We use finite element analysis to convert the spherically symmetric distribution into a uniform distribution of charge points and locate the images below the conductive plane by the usual method. Then we let the number of charge points go to infinity. We will show that the simple model (which assumes that the spherically symmetric charge distribution can be replaced by a point charge in the middle of the sphere even when it is in contact with a conductive plane) underestimates the force of adhesion because it ignores the force due to the charges in the proximity of the conductive plane.

2. Theory

We model a spherically symmetric charge distribution using a finite element analysis both analytically and with numerical calculations. The procedure allows us to substitute a continuous function of surface charge density with a set of discrete charge points such that the total charge magnitude is conserved. This procedure is well known in the field of finite element methods, where distributed body forces, tractions (surface forces distributed over a finite area such as frictional or constrain forces), and masses are allocated among the nodes of the elements. Among many different possible distribution techniques, there is a particular arrangement of charge points along a set of annuli parallel to the conductive plane that facilitates a derivation of a closed form solution for the electrostatic forces. In Fig. 1 the total number of annuli on the sphere is N. The dots on the annuli show the arrangements of charge points on a sphere of radius R resting on the conductive plane at $z = 0$. The charge points are chosen using polar coordinates to maintain a constant arc length $R\Delta\phi$ between charge points in the two orthogonal directions. Therefore, the vertical angle between two adjacent annuli, $\Delta\phi$, is simply given by π/N. Since, the circumference of each annulus is different, they each hold a different number of charge points. The number of charge points for the ith annulus is given by

$$k_i = 2N\sin(\pi\, i/N + \pi/2N), \quad i = 0, \ldots, N-1, \tag{1}$$

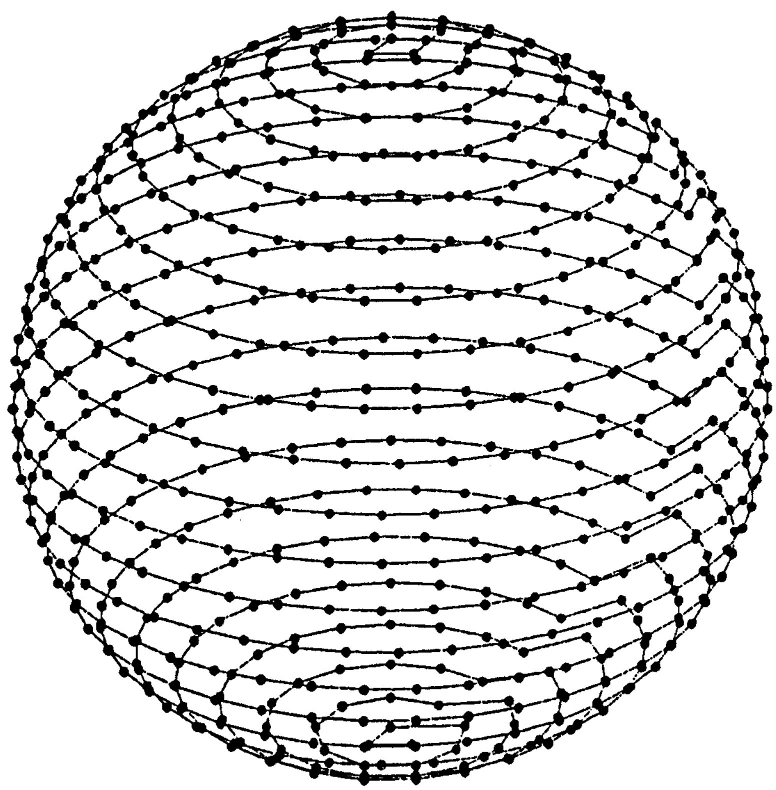

Fig. 1. Sphere with points of charge.

where the expression in parenthesis is the latitude angle of annulus i. Eq. (1) is derived by dividing the circumference ($2\pi R\sin(i\Delta\phi + \Delta\phi/2)$) by the arc length $R\Delta\phi$ where $\Delta\phi = \pi/N$. The total number of charge points, K can be derived by summing of the charge points all of the annuli

$$K = \sum_{i=0}^{N-1} k_i = 2N\int_0^N \sin(\pi x/N)\,\mathrm{d}x \approx \frac{4N^2}{\pi} \tag{2}$$

or by dividing the total surface area of the sphere by the area that a charge point occupies $(R\pi/N)^2$. The charge q in each charge point is the total charge on the sphere Q divided by the number of charge points K

$$q = \frac{Q}{K} = \frac{Q\pi}{4N^2}. \tag{3}$$

For example, a sphere of charge $Q = 12$ fC that is subdivided with $N = 90$ annuli will have $K = 10,313$ charge points, each with a charge $q = Q/K = 1.16\times10^{-18}$ C, or about 7 electrons per charge point.

The number of charge points on the first annulus nearest the conductive plane is given, in the limit for a large N, by

$$k_0 = 2N\sin(\pi/2N) \approx \pi. \tag{4}$$

The plane of the ith annulus crosses the z-axis at

$$z_i = R\left[1 - \cos\left(\frac{\pi}{2N} + \frac{\pi i}{N}\right)\right], \quad i = 0, \ldots, N-1. \tag{5}$$

Using the first two terms of the Taylor series expansion, the separation, z_0 of the first annulus from the reference plane $z = 0$ is given by

$$z_0 = \frac{R}{2}\left(\frac{\pi}{2N}\right)^2. \tag{6}$$

(For $R = 6$ μm and $N = 180$, $z_0 = 2.3$ Å, which is as close as two materials are ever assumed to approach each other [7].)

Consider the electrostatic forces due to the interactions between the charge points located in proximity to the conductive plane (which are on the first annulus) with their image charges located symmetrically across the conductive plane. Using Coulomb's law, the force on a single charge point q in the first annuli due to its own image charge point F_{11} (the attractive forces can be expressed by the elements of a rank 2 matrix $[F_{ij}]$, where index i refers to a charge point in an annuli and index j refers to an image charge point) located symmetrically across the conductive plane can now be derived, using Eqs. (3) and (6), in a closed form by

$$F_{11} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z_0)^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2R)^2}\frac{4}{\pi^2}. \tag{7}$$

The functional dependence of q and z_0 on the number of annuli N as given by Eqs. (3) and (6) cancels out. Since there are approximately π charge points in the first annuli (Eq. (4)), the contribution to the electrostatic force by these charges, which we will call the proximity force F_p, is

$$F_p = \pi F_{11} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2R)^2}\frac{4}{\pi}. \tag{8}$$

This is a remarkable result since only a few point charges with a fractional charge q, which are located in the vicinity of the contact point can generate an attractive force $1.27(4/\pi)$ times greater than a charge point of charge Q located at the center of sphere. We note that this result is independent of the number of annuli, N, in the limits of large N, which suggests that this result is independent of the particular charge distribution chosen for this calculation.

In principal, all of the other image charges contribute to forces on the charges in the first annuli, which are identified as F_{12}, F_{13}, etc. However, for a large N, the separation $2z_0$ is small compared to the distance to all other charge points. The closest image charges are those due to the charges in the second annuli (z_1) which are easily shown to have negligible contributions since $z_1/z_0 = 9$.

Since the number of point charges considered in the proximity annulus is much smaller than the total number of charge points, i.e. $k_0 \ll K$, the rest of the charge points can still be considered as a complete sphere of charge. This can be modeled by the usual method of placing a single charge in the center of the sphere, giving for the force for the bulk charge F_b

$$F_b = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2R)^2}. \tag{9}$$

The sum of the electrostatic forces due to the bulk of charges Q from Eq. (9) and due to the proximity charges from Eq. (8), is

$$F = F_b + F_p = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2R)^2}\left(1 + \frac{4}{\pi}\right) \tag{10}$$

which has a new factor $(1 + 4/\pi)$. This is 2.27 times greater than the force calculated by the simple image model. This simple derivation shows that a closed form solution for the electrostatic force of a spherically symmetric charge distribution in contact with a conductive plane can be derived in a straightforward way and provides a useful and universal result.

3. Numerical Calculations

A numerical calculation was carried out to support the analytical derivation of the electrostatic image forces. As before, the assumption is made that the charge on the surface of a uniformly charged sphere can be lumped with K point charges. Two parameters were varied, the number of annuli N and the separation s between the bottom of the sphere and the conductive plane. The force of attraction is given by the double sum of the force of attraction of all the charge points to all of the image charges

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\sum_{i}^{K}\sum_{j}^{K} q_i q_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}. \qquad (11)$$

Index i is assigned to the point charges and the index j assigned to the image point charges. The position vectors **r** define the location of the charge points from the center of a Cartesian coordinate system coinciding with the contact point. The image force F in Eq. (11) can be normalized with the bulk force F_b, giving F/F_b which we will call the correction factor. The correction factor is plotted against the separation distance s in Fig. 2 for the number of annuli $N = 40$, 90 and 180. At a separation distance s larger than 50 nm, the electrostatic force exerted on a spherically symmetric sphere of charges is well represented by the conventionally assumed value (Eq. (9)), i.e. the correction factor is equal to one. On the other hand, at a separation distance s smaller than 50 nm, Eq. (9) does not equal the total electrostatic force and the ratio F/F_b diverges from the value of 1 depending on the number of annuli N, which determines the precise locations of the point charges. However, all force curves converge to the same value $(1 + 4/\pi)$ at contact, independent of the number of annuli. This result is in excellent agreement with the general analytical result, Eq. (10).

Note from Fig. 2 that values of s at which this new proximity force is active depends on the assumptions in the model, i.e. the number of annuli assumed. This suggests that the observation of this force as a function of gap will depend on the details of the actual charge distribution in the proximity of the contact. If we assume the proximity force can be observed when it is 10% of the bulk force, then it is easily shown (by replacing z_0 in Eq. (7) with $z_0 + s_{10}$ and equating π times the force to 0.1

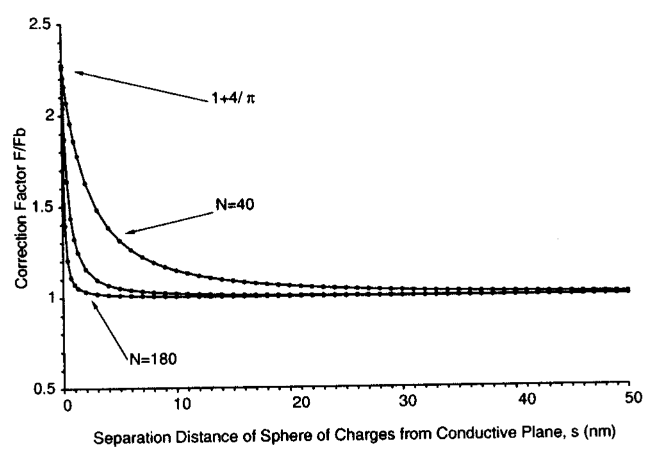

Fig. 2. Correction factor to the electrostatic force normalized to F_b vs. separation distance s between the sphere and the conductive plane, and N, the number of annuli, for a 6 μm radius sphere. The curves are for N =40, 90, and 180.

times Eq. (9)) that the gap s_{10} at this point is

$$s_{10} = \left(\sqrt{\frac{40}{\pi}} - 1\right)\frac{R\pi^2}{8N^2}. \tag{12}$$

Using Eq. (2), Eq. (12) becomes

$$s_{10} = \frac{R}{K}\left(\sqrt{\frac{40}{\pi}} - 1\right)\frac{\pi}{2} = 4.0\frac{R}{K}. \tag{13}$$

Noting that charge is quantized so that K cannot be larger than Q/e, where e is the electronic charge, the minimum s_{10} is $4Re/Q$. This is 3.2 Å for a fully and uniformly charged electrophotographic toner particle ($Q = 12$ fC and $R = 6$ μm, which is about 12 μC/g) but is 320 Å for a particle charged with clusters of 100 electrons per charge point.

4. Discussion

Assuming that a spherically symmetric charge distribution can be modeled as a set of distributed charge points, we have shown that the electrostatic force on the sphere in contact with a conductive plane has two components: (1) one due the bulk charges which can be calculated by replacing the sphere with its charge in its center and (2) a newly derived component due to the charges in proximity to the plane. The proximity force is larger than the bulk force component by a factor $4/\pi$. The total electrostatic force acting on a sphere with a spherically symmetric charge distribution in contact with a conductive plane has a correction factor $(1 + 4/\pi)$ as compared with the conventionally used formula.

We have shown this result both analytically and numerically. In both cases the magnitude of the force calculated when the sphere contacts the plane is in excellent agreement for sufficiently large number of annuli. The fact that our result is independent of the details of the model, i.e. the number of annuli, suggests that this result is general. Further, we can physically identify the source of the proximity force: it is due to the charges in closest proximity to the contact point.

In contrast, the gap s at which this new proximity force is active depends on the number of annuli assumed, i.e. the details of the model. This suggests that the observation of the proximity force will depend on the details of the charge distribution in the vicinity of the contact. The shape of the force function in the pre-contact region for a separation distance less than 50 nm may give an important clue about the magnitude and the distribution of charges in the proximity of the contact point. For example, a sharp increase of force by a factor $1 + 4/\pi$ within a few nm prior to contact (consistent with Eq. (13)) might indicate a very uniform surface charge density. Alternatively, an experimental observation of a force increase at much larger separations might be interpreted as patches of charge which are multiples of the elementary electronic charge.

This new force may account for observations of forces on particles observed in force microscope experiments in which an unexpected and unexplained force of shorter range than electrostatic but longer range than van der Waals was observed [4–6]. For example, in one of these experiments a charged polystyrene sphere was brought in close proximity to a conducting plane [5]. At close distances, the force observed could be quantitatively accounted for by van der Waals force. But beyond 10 nm, the force observed was much larger than could be accounted for by van der Waals force but was also much larger than could be accounted by the standard electrostatic image force. The proximity force proposed here can account for the force in this region [4]. In Refs. [5,6] it is suggested that the unexpected force is due to a localized charge patch near the point of contact. However, as the authors themselves point out this leads to a dilemma (in their words): the electric field due to the postulated localized charge patch exceeds the electric field that air can support by an enormous factor, roughly 170. Accounting for the data with the proximity force eliminates this dilemma.

Extensive discussions exist in the literature attempting to identify whether the dominant force of adhesion on insulating particles is due to electrostatic forces, with uniform or nonuniform surface charge distributions, or van der Waals forces [2,3]. We suggest that insulating particle adhesion in general, and toner particle adhesion (in electrophotography) in particular may be largely due to the proximity force. Consider that real particles are not perfect spheres. They have many contact points with a conductive plane. Assume that at every contact point the proximity force is active. In this case the proximity force could dominate the other forces of adhesion that have so far been considered in the literature and therefore may be able to explain the large toner adhesion forces reported in the literature [2] without the need to assume nonuniform charge distribution or unusually large van der Waals forces [8].

Acknowledgements

The authors would like to acknowledge Bobo Wang for support and encouragement, Randy Sanders, Jack Pei, and Dave Stockman for their encouragement, and Randy Sanders and Joe Crowley for comments on the manuscript.

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